Norms on Categories and Generalized Schröder-Bernstein Theorems

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Metrizations

Motivation

* geometric sturcture on collection of objects
* quantitative statements
* fixed point theorems
* etc.

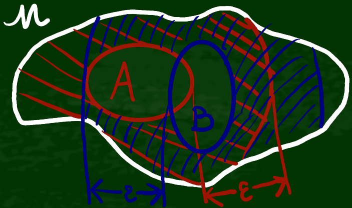
Easy example

Define the **Hausdorff distance** $\mathrm{d_H}(A, B)$ of closed subsets A, B of a metric space $\mathcal{M} = (M, d_\mathcal{M})$ as

$$\inf\{\, \varepsilon > 0 \mid A^\varepsilon \subset B \text{ and } B^\varepsilon \subset A \,\}$$

where A^ε is the ε-thickening

$$A^\varepsilon := \{\, x \in M \mid \exists y \in A\colon |x\,y| \le \varepsilon \,\}.$$



More complicated ones

Gromov-Hausdorff distance $\mathrm{d_{GH}}(\mathcal{M}, \mathcal{N})$ of two (isometry classes of) metric spaces $\mathcal{M} = (M, d_\mathcal{M}), \mathcal{N} = (N, d_\mathcal{N})$

$$\inf_{\mathcal{M} \xrightarrow{\varphi} \mathcal{L} \xleftarrow{\psi} \mathcal{N}} \mathrm{d_H}(\varphi_!(M), \psi_!(N))$$

where φ, ψ are embeddings and $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ image map for $f\colon X \to Y$.

Pointed Gromov-Hausdorff distance between $(M, d_\mathcal{M}, p)$ and $(N, d_\mathcal{N}, q)$

$$\sum_{n=1}^{\infty} \inf_{r,s \ge n} \mathrm{d_{GH}}(\mathrm{B}[p, r], \mathrm{B}[q, s])$$

where $\mathrm{B}[p, r]$ is the closed r-ball around p.

Put measure on top
$\rightsquigarrow$ pointed metric measure space

Clear it works, but

* technical details matter,
* axioms & basic properties hard to check,
* different ways
* including arbitrary choices.

Idea

Aim

Easy and systematic metrizations.

First Observation

Recall: $\mathrm{d_H}(A, B) = \inf\{\,\varepsilon > 0 \mid A^\varepsilon \subset B, B^\varepsilon \subset A\,\}$.

Symmetrized!

Things get easier when dropping the symmetry requirement

$\rightsquigarrow$ quasimetric

Second observation

Nicest metrics are induced by norms.

Recall Axioms of $\|.\|$:

(i) $\|.\|$: ~~vector~~ space $\to [0, \infty$~~)~~$]$, some algebraic structure

(ii) ~~$\|\lambda.v\| = \lambda\|v\|$,~~

(iii) triangle inequality $\|v + w\| \leq \|v\| + \|w\|$ } when "v+w" is defined

(iv) $\|v\| = 0$ ~~$\iff$~~ $v = 0$ more complicated

The priordial example

Sets and functions.

(i) $\|.\|$: (functions, composition) $\to [0, \infty]$,

$$\|f\|_{\text{set}} := \log \sup^1_{y\in Y} \#f^*(\{y\}),$$

where

$f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ preimage,

$$\sup^a_{x\in X} f(x) := \{a\} \cup \{\, f(x) \mid x \in X\,\}.$$

(iii) for $X \xrightarrow{f} Y \xrightarrow{g} Z$:

$$\begin{aligned} \|g \circ f\|_{\text{set}} &= \log \sup^1_{z\in Z} \#f^* g^*(\{z\}) \\ &\leq \log \sup^1_{z\in Z} \Big(\sup^1_{y\in Y} \#f^*(\{y\}) \cdot \#g^*(\{z\})\Big) \\ &= \|f\|_{\text{set}} + \|g\|_{\text{set}}, \end{aligned}$$

(iv) Observe $\|f\|_{\text{Set}} = 0 \iff f$ is injective.

Schröder-Bernstein theorem: Given $f\colon X \to Y$ and $g\colon Y \to X$, we have

$$\|f\|_{\text{Set}} = \|g\|_{\text{Set}} = 0 \implies \exists \text{ bijection } X \to Y$$

Brief introduction to category theory

Observation

In a mathematical theory one often deals with a specific class of objects and maps between them, e.g.

* sets & functions,
* topological spaces & continuous maps,
* groups & group homomorphisms,
* normed vector spaces & bounded linear maps,
* etc.

Definition

A category $\underline{C} = (\underline{C}_0, \underline{C}_1, ;, \mathrm{id})$ comprises

* $\underline{C}_0$ class of objects,
* $\underline{C}_1$ class of morphisms,
* $\mathrm{source}\colon \underline{C}_1 \to \underline{C}_0$, $\mathrm{target}\colon \underline{C}_1 \to \underline{C}_0$,
* composition: $(f, g) \mapsto f \,;\, g$ defined if $\mathrm{target}\, f = \mathrm{source}\, g$, (notation $f\colon X \to Y$ iff $X = \mathrm{source}\, f$ and $Y = \mathrm{target}\, f$),
* $X \mapsto \mathrm{id}_X$ for $X \in \underline{C}_0$.

such that

* $\mathrm{source}(f \,;\, g) = \mathrm{source}\, f$, $\mathrm{target}(f \,;\, g) = \mathrm{target}\, g$;
* $\mathrm{id}_X\colon X \to X$;
* $\mathrm{id}_X \,;\, f = f$ and $g \,;\, \mathrm{id}_X = g$;
* ; is associative.

Notation $\underline{C}[X, Y] := \{ f \mid f\colon X \to Y \}$

Notation $\underline{\mathrm{Set}}$, $\underline{\mathrm{Top}}$, $\underline{\mathrm{Gr}}$, $\underline{\mathrm{NVect}}$, etc.

Notation $g \circ f := f \,;\, g$. In $\underline{\mathrm{Set}}$ we have $(g \circ f)(x) = (f \,;\, g)(x) = g(f(x))$.

$f\colon X \to Y$ is an **isomorphism** $\iff$ f is invertible, i.e. $\exists (g\colon Y \to X)$: $\mathrm{id}_X = f \,;\, g$ and $\mathrm{id}_Y = g \,;\, f$

Examples

* $\underline{C}_0$ is a singleton: monoid. If additionally every morphism is an isomorphism: group.
* $\underline{C}[X, Y]$ is empty or a singleton for every $X, Y \in \underline{C}_0$: preorder.

The Axioms

Definition: seminorm

A **seminorm** on a category $\underline{C}$ is a map $\|.\|\colon \underline{C}_1 \to [0,\infty]$ such that

(N1) $\|\mathrm{id}_X\| = 0$ for all $X \in \underline{C}_0$;

(N2) $\|f\,;g\| \le \|f\| + \|g\|$ (triangle inequality).

Definition: norm

A seminorm such that for all $X, Y \in \underline{C}_0$

(N3) if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ with $\|f\| = \|g\| = 0$, then there is an isomorphism $X \to Y$;

(N4) if for all $\varepsilon > 0$ there is $f\colon X \to Y$ with $\|f\| \le \varepsilon$, then there is $f\colon X \to Y$ with $\|f\| = 0$.

Back to priordial example

For $(\underline{\mathrm{Set}}, \|.\|_{\mathrm{Set}})$, remember

$$\|f\|_{\mathrm{set}} = \log \sup_{y\in Y}{}^{1} \#f^*(\{y\}),$$

axioms (N1-3) are already checked.

(N4) follows from the fact that $\|.\|_{\mathrm{Set}}$ is discretely valued.

Metrization

Let $(\underline{C}, \|.\|)$ be a seminormed category.Let $\mathrm{sk}_0(\underline{C}, \|.\|)$ denote the set of isomorphism classes of objects in $\underline{C}$.

Define the quasipseudometric

$$d_{\|.\|}(\hat{X}, \hat{Y}) := \inf\{\,\|f\| \mid f\colon X \to Y\,\}$$

where $X \in \hat{X}$ and $Y \in \hat{Y}$.

By some symmetrization, e.g.

$$d^{\vee}_{\|.\|}(X,Y) := d_{\|.\|}(X,Y) \vee d_{\|.\|}(Y,X) \quad \text{or}$$

$$d^{+}_{\|.\|}(X,Y) := \tfrac{1}{2}\left(d_{\|.\|}(X,Y) + d_{\|.\|}(Y,X)\right)$$

we obtain a pseudometric.

Imposing axiom (N3), this pseudometric becomes a metric.

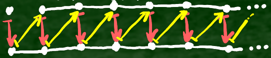
Examples

Let Graph denote the category with

objects graphs (V, E), where
$V = \{\text{vertices}\}$ and
$E = \{(\text{undirected}) \text{ edges}\}$.

morphisms $(V, E) \to (V', E')$ function
$f\colon V \to V'$ such that
$(v, w) \in E \implies (f(v), f(w)) \in E'$.

$(\text{Graph}, \|.\|_{\text{Set}})$ is seminormed but not normed, e.g.



But restricting Graph to finite graphs $\|.\|_{\text{Set}}$ becomes a norm:
If V, V' have same cardinality, every injection between them is a bijection.
Hence $f_1\colon (v, w) \mapsto (f(v), f(w)$ is an injection $E \to E'$. So is
$g_1\colon (v, w) \mapsto (g(v), g(w)$.
Thus f_1 is a bijection.

In non-discrete examples the role of finiteness will be played by compactness.

Let $\text{NVect}^*_{\mathbb{R}}$ denote the category of normed vector spaces over the reals and linear maps.

$$\|A\|_{\text{op}} := \log \sup_{v \in V}{}^{1} \frac{\|v\|_V}{\|Av\|_W}$$

If $\|A\| = 0$, then A is called **expansive**.
$\|A\|_{\text{op}}$ is not a norm on $\text{NVect}^*_{\mathbb{R}}$. But on ${}^{\text{Hilb}}\text{NVect}^*_{\mathbb{R}}$, the subcategory of Banach spaces that admit an inner product, i.e. a Hilbert space structure. This is because two Hilbert spaces are isomorphic iff they have the same Hilbert space dimension.

Question

Is there a connection to the more intuitive seminorm

$$\log \sup_{v \in V}{}^{1} \frac{\|Av\|_W}{\|v\|_V}$$

The **left dual seminorm** of $\|.\|$

$$\|f\|^{*\mathrm{L}} := \sup_{f'}{}^{0} \left(\|f'\| - \|f' \,;\, f\|\right)$$

where $X \xrightarrow{f'} Y \xrightarrow{f} Y'$.

Precapacities

Definition: functor

Functors are the structure preserving morphisms between categories, i.e. $F\colon \underline{C} \to \underline{D}$ consists of two functions

$$F_0\colon \underline{C}_0 \to \underline{D}_0,\ F_1\colon \underline{C}_1 \to \underline{D}_1$$

that are compatible with ;, source, and target, i.e. ....

$\rightsquigarrow \underline{\mathrm{Cat}} = (\{\text{categories}\}, \{\text{functors}\})$.

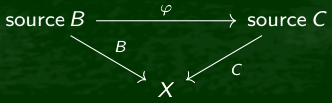
Concrete category

A **concrete category** is a faithful functor $F\colon \underline{C} \to \underline{\mathrm{Set}}$. A functor is **faithful** if F_1 is injective on $\underline{C}[X, Y]$.

Subobjects

$X \xrightarrow{f} Y$ is **monomorphism** if $\forall h_1, h_2\colon X' \to X$: $h_1 \,;\, f = h_2 \,;\, f \implies h_1 = h_2$.

A **subobject** of X is an equivalence class of monomorphisms into X. They form a partially ordered class $(\mathrm{Sub}(X), \subseteq)$:



By a **concrete category with subobjects** $(\underline{C}, F, \mathrm{SO})$ we understand a concrete category $(\underline{C}, F)$ additionally endowed with a selection function

$$\mathrm{SO}\colon X \mapsto \mathrm{SO}(X),$$

where $X \in \underline{C}_0$, $\mathrm{SO}(X) \subseteq \mathrm{Sub}(X)$, such that preimages of subobjects in $\mathrm{SO}(Y)$ are well-defined, i.e. for all $f\colon X \to Y$ and $C \in \mathrm{SO}(Y)$ $\exists!$ maximal $B \in \mathrm{SO}(Y)$ with $|B| = (Ff)^*(|C|)$ where $|C| := (FC)(\text{source } C) \subseteq F(X)$. Write $f^*C := B$.

Precapacity

A **precapacity** c on $(\underline{C}, F, \mathrm{SO})$ is a

$$\{\, C \in \mathrm{SO}(X) \mid X \in \underline{C}_0 \,\} \xrightarrow{c} [0, \infty]$$

that is monotone, i.e. for any $B, C \in \mathrm{SO}(X)$: $B \subseteq C \implies c(\text{source } B) \leq c(\text{source } C)$.

$$\|f\|_c := \sup_{\substack{C \in \mathrm{SO}(Y),\\ c(C) < \infty}}{}^{0} \; c(f^*C) - c(C)$$

defines a seminorm.

Topological spaces

Let $\mathcal{X} = (X, \tau_{\mathcal{X}})$ denote a top. space. Define the **fiber dimension seminorm**

$$\|f\|_{\text{f dim}} := \|f\|_{|\log(1+\dim)|} = \sup^0_{\substack{A \in \mathcal{P}(Y) \\ \dim A < \infty}} |\log(1 + \dim f^* A)| - |\log(1 + \dim A)|.$$

Define $I(\mathcal{X}) := \{\text{connected components}\}$ and the **disconnectedness seminorm**

$$\|f\|_{\text{disconn}} := \|f\|_{|\log \# I|} = \sup^0 \left\{ |\log(\#(I\, f^* C))| - \log(\#(I\, C)) \;\middle|\; \substack{C \subset Y \text{ closed}, \\ 0 < \#(I\, C) < \infty} \right\}.$$

Fiber-wise characterization

By so-called Hurewicz formula

$$\|f\|_{\text{f dim}} = \sup_{y \in Y} |\log(1 + \dim(f^* y))|$$

for a map

$f\colon (\mathrm{T}_4\text{-space}) \to (\text{metrizable space})$.

For $\|f\|_{\text{disconn}}$ we have in general

$$\|f\|_{\text{disconn}} = \sup^0_{\substack{C \neq \emptyset \text{ closed}, \\ \# I(C) = 1}} |\log(\#(I(f^* C)))|$$

and, if $\mathcal{Y}$ is a T_1,

$$\|f\|_{\text{disconn}} \geq \sup^0_{p \in Y} |\log(\#(I\, f^*\{p\}))|.$$

Characterization of $\|.\| = 0$

$f\colon \mathcal{X} \to \mathcal{Y}$ is

light if the fiber $f^*\{y\}$ is totally disconnected for every $y \in Y$, i.e. when $\dim f^*\{y\} = 0$,

monotone if the preimage of every $\{y\} \subset \mathcal{Y}$ is nonempty and connected.

Schröder-Bernstein theorem

$$\|f\|_{\text{top}} := \|f\|_{\text{disconn}} + \|f\|_{\text{f dim}}$$

Let $\mathcal{X}$ be compact, T_4 and $\mathcal{Y}$ be metrizable. Then $\|f\|_{\text{top}} = 0 \implies f$ is a homeomorphism.

Especially, on the category of compact metrizable spaces $\|.\|_{\text{top}}$ is a norm.

Metric spaces

$\underline{\text{Met}} := (\{\text{metric spaces}\}, \{\text{functions}\})$.

dilatation seminorm

$$c(A) := \operatorname{diam}(A) = \sup_{x,y\in A}{}^0 |x\,y|,$$

$$\|f\|_{\text{dil}} := \|f\|_c$$
$$= \sup_{A\subseteq N}{}^0 (\operatorname{diam}(f^*A) - \operatorname{diam}(A))$$
$$= \sup_{x,y\in M}{}^0 (|x\,y| - |f(x)\,f(y)|).$$

measuring deviation from being expansive

$\operatorname{diam} M = \|M \to T\|_{\text{dil}}$, where $T = (\{\bullet\}, 0)$ is terminal object.

Left dual

$$\|f\|^{*\text{L}}_{\text{dil}} = \sup_{x,y\in M}{}^0 (|f(x)\,f(y)| - |x\,y|)$$

deviation from being a contraction.

When treating metric spaces in category theory, one normally restricts attention to contractions, though in metric space theory all kinds of maps are considered.

Let $\underline{\text{Met}}_{\text{cpt}}$ be $\underline{\text{Met}}$ restricted to compact spaces.

Gromov-Hausdorff distance

$((\underline{\text{Met}}_{\text{cpt}})_0, \mathrm{d}_{\text{GH}}) \to ((\underline{\text{Met}}_{\text{cpt}})_0, \mathrm{d}^+_{\text{dil}})$ is 4-Lipschitz with continuous inverse.

$\|.\|_{\text{dil}}$ is a norm on $\underline{\text{Met}}_{\text{cpt}}$.

Lemma/(N4)

Let $\mathcal{M}$ be a compact. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $h\colon \mathcal{M} \to \mathcal{M}$ with $\|h\|_{\text{dil}} < \delta$ implies that

(i) $h(M)$ is ε-dense, and

(ii) $\|h\|^{*\text{L}}_{\text{dil}} \leq 4\varepsilon + C\delta$ where $C = C(\varepsilon, \mathcal{M})$.

Lipschitz seminorm

$$\|f\|_{\text{Lip}} := \|f\|_{\log\operatorname{diam}}$$
$$= \sup_{x,y} \frac{|x\,y|_{\mathcal{M}}}{|f(x)\,f(y)|_{\mathcal{N}}}$$

Further directions

metric measure spaces, limits, etc.